Math 254-2 Exam 11 Solutions

1. Carefully define the term "basis". Give two examples in $\mathbb{R}^2$.

   A basis is a set of vectors that is both independent and spanning. Equivalently, it is a set of vectors that is maximal and independent. Equivalently, it is a set of vectors that is minimal and spanning. Examples in $\mathbb{R}^2$ include $\{(1,0),(0,1)\}$ and $\{(1,0),(1,1)\}$.

For the remaining problems, consider the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.
Hint: all solutions can be expressed with integers.

2. Calculate the characteristic polynomial $\Delta(t)$ (or $p(\lambda)$) for A.

   $p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-2 & 1 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(-1)^{2+2} \begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = (\lambda-1)((\lambda-2)^2-1) = (\lambda-1)(4-4\lambda+\lambda^2-1) = (\lambda-1)(\lambda^2-4\lambda+3) = (\lambda-1)(\lambda-3)(\lambda-1) = \lambda^3 - 5\lambda^2 + 7\lambda - 3$.

3. Find all the eigenvalues of A.

   We solve $0 = p(\lambda) = (\lambda-3)(\lambda-1)^2$. It has two solutions: $\lambda = 3$ and $\lambda = 1$ (a double root). Hence $1, 3$ are the eigenvalues of A.

4. For each eigenvalue, find a maximal independent set of eigenvectors.

   For $\lambda = 1$, $A - \lambda I = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. This has row canonical form $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since there are two pivots, the rank is two and the nullity is one; hence the eigenspace is one-dimensional. If $(x,y,z)^T$ is in the nullspace, then $x + z = 0, y = 0$. Hence a basis for the eigenspace is $(1,0,-1)^T$.

   For $\lambda = 3$, $A - \lambda I = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. This has row canonical form $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since there are two pivots, the rank is two and the nullity is one; hence the eigenspace is one-dimensional. If $(x,y,z)^T$ is in the nullspace, then $x - z = 0, y = 0$. Hence a basis for the eigenspace is $(1,0,1)^T$.

5. For each eigenvalue, give its algebraic and geometric multiplicity. Is A diagonalizable? What is the Jordan form of A?

   $\lambda = 3$ is a single root, hence its algebraic multiplicity (and thus geometric multiplicity) is 1. $\lambda = 1$ is a double root, hence its algebraic multiplicity is 2. However, its geometric multiplicity is 1, as calculated in the previous problem. Hence A is not diagonalizable, and it has Jordan form $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ or $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.